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SELF-ADJOINT DIFFERENTIAL EQUATIONS ARISING IN FINITE ELASTICITY FOR SMALL SUPERIMPOSED DEFORMATIONS

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Abstract—For a number of problems involving small deformations superimposed upon large non-homogeneous deformations of isotropic incompressible hyperelastic materials the governing fourth order linear ordinary differential equations are shown to be self-adjoint. Moreover it is shown that at least in principle every fourth order linear self-adjoint differential equation can be factorized in terms of a single second order self-adjoint operator. These two results unify the derivation of a number of solutions of these equations which have been derived previously by a variety of *ad-hoc* procedures. Two problems are posed for the interested reader. Firstly the problem of obtaining these self-adjoint differential equations directly from the underlying variational principle and secondly the problem of obtaining the explicit factorizations for the differential equations given.

1. INTRODUCTION

Stability investigations in finite elasticity involving large non-homogeneous deformations of isotropic incompressible hyperelastic materials give rise to inhomogeneous fourth order linear ordinary differential equations for the small superimposed deformation. These equations are in general extremely complicated and although a number of special solutions are known (see for example Hill[1–4]) there are as yet no general solutions applicable to all materials and all buckling modes. The purpose of this note is to point out that these differential equations are self-adjoint and that at least in principle, every fourth order linear self-adjoint differential equation can be factorized in terms of a single second order self-adjoint operator. In practice the actual determination of this second-order operator is not always apparent and we formulate this problem for the interested reader. Because of the underlying variational principle in finite elasticity, it is not altogether surprising that these fourth order differential equations are self-adjoint, although the actual derivation of these self-adjoint forms directly from the variational principle is by no means obvious and we also pose this problem.

In the following section we give the decomposition of the general fourth order linear self-adjoint differential equation in terms of a second order self-adjoint operator. This decomposition enables all four solutions of the fourth order equation to be expressed in terms of the two linearly independent solutions of the second order differential equation. Also in this section we note a slightly stronger decomposition of fourth order self-adjoint equations, which includes the first decomposition as a special case.

In Sections 3–5 we give the appropriate fourth order self-adjoint differential equations for a number of problems involving large non-homogeneous deformations of isotropic incompressible hyperelastic materials. In Section 3 for the problem of the stability of thick-walled cylindrical tubes under external pressure, we show that the solutions given in [1 and 3] can be obtained from the results given in Section 2. Also in Section 3 we show that the corresponding dynamical problem (see Wang and Ertepinar[5]) and the length wise buckling problem of a cylindrical tube also give rise to fourth order self-adjoint differential equations. The solutions given in [2 and 4] for the problem of the stability of a thick-walled spherical shell under external pressure are shown in Section 4 to be obtained from the self-adjoint form of the fourth order differential equation and the results given in Section 2. In Section 5 we consider small deformations superimposed upon the plane straightening and stretching of a sector of a circular-cylindrical tube and we show that by a change of the independent variable, the governing fourth order equation can be made self-adjoint. The solutions given in [1 and 2] are applicable only for special materials for which the fourth order equations are greatly simplified. The deformation given in Section 5 provides an example for which there is apparently no material giving corresponding simplifications.

Although the observation of self-adjointness has not so far led to any new solutions of the governing equations, it does provide some symmetry to what are otherwise extremely complicated problems. Moreover it evidently relates a number of standard results, such as orthogonality of eigenfunctions and Rayleigh's quotient for approximating eigenvalues, to problems in finite elasticity. In addition it could well provide alternative numerical procedures for solving equations of this type.

2. FOURTH ORDER LINEAR SELF-ADJOINT ORDINARY DIFFERENTIAL EQUATIONS

The general fourth order linear self-adjoint differential equation for $u(R)$ is given by (see for example Murphy[6], p. 200)

$$[f_0(R)u'']'' + [f_1(R)u']' + f_2(R)u = 0, \tag{2.1}$$

where $f_0(R)$, $f_1(R)$ and $f_2(R)$ are any sufficiently differentiable and continuous functions of R and primes denote differentiation with respect to R. We define the second order self-adjoint differential operator D^2 by

$$D^2u = [a(R)u']' + b(R)u, \tag{2.2}$$

where $a(R)$ and $b(R)$ are arbitrary functions of R. If $c(R)$ is another arbitrary function of R then we can show that the fourth order equation

$$D^2[c(R)D^2u] = 0, \tag{2.3}$$

is self-adjoint where the functions $f_0(R)$, $f_1(R)$ and $f_2(R)$ are given by

$$\begin{aligned} f_0(R) &= a^2c,\\ f_1(R) &= a(ca')' + 2abc,\\ f_2(R) &= b^2c + [a(bc)']'. \end{aligned} \tag{2.4}$$

Thus if $f_0(R)$, $f_1(R)$ and $f_2(R)$ are known functions of R then (2.4) constitutes three equations for the determination of the three unknown functions $a(R)$, $b(R)$ and $c(R)$ and consequently at least in principle every fourth order self-adjoint eqn (2.1) can be expressed in the form (2.3). In practice however the actual determination of $a(R)$, $b(R)$ and $c(R)$ from (2.4) may be difficult. If we use $(2.4)_1$ and $(2.4)_2$ to obtain expressions for $c(R)$ and $b(R)$ in terms of $a(R)$ then on substituting these expressions in $(2.4)_3$ we obtain a highly non-linear fourth order ordinary differential equation for $a(R)$. The resulting equation is evidently considerably more complicated than the original eqn (2.1) so that the above observation is advantageous only when (2.4) admits simple solutions for $a(R)$, $b(R)$ and $c(R)$ which can be obtained by inspection. We remark that since $a(R)$ is obtained as a solution of a fourth order differential equation, the factorization of (2.1) in the form of (2.3) is not unique.

For the time being we suppose that solutions of (2.4) can be found and we let $u_1(R)$ be a solution of the second order equation

$$D^2u = 0. \tag{2.5}$$

A second linearly independent solution $u_2(R)$ of (2.5) is given by

$$u_2(R) = u_1(R)\int_{R_0}^{R}\frac{\mathrm{d}t}{a(t)u_1(t)^2}, \tag{2.6}$$

where R_0 is any arbitrary constant and the Wronskian $\omega(R)$ of these linearly independent solutions is given by

$$\omega(R) = u_2u_1' - u_1u_2' = -\frac{1}{a(R)}. \tag{2.7}$$

From (2.3) we have

$$D^2u = -\frac{[\gamma_1 u_1(R) + \gamma_2 u_2(R)]}{c(R)}, \tag{2.8}$$

where γ_1 and γ_2 are arbitrary constants and by the usual variation of parameters method, it is a simple matter to obtain the general solution of (2.3) as

$$\begin{aligned} u(R) = {} & \gamma_1 \left\{ u_1(R) \int_{R_1}^{R} \frac{u_1(t)u_2(t)}{c(t)}\,dt - u_2(R) \int_{R_1}^{R} \frac{u_1(t)^2}{c(t)}\,dt \right\} \\ & + \gamma_2 \left\{ u_1(R) \int_{R_1}^{R} \frac{u_2(t)^2}{c(t)}\,dt - u_2(R) \int_{R_1}^{R} \frac{u_1(t)u_2(t)}{c(t)}\,dt \right\} \\ & + \gamma_3 u_1(R) + \gamma_4 u_2(R), \end{aligned} \tag{2.9}$$

where γ_3, γ_4 and R_1 are further arbitrary constants where R_1 is chosen so as to avoid any singularities in the integrands and need not necessarily be the same for each of the integrals in (2.9). Evidently if the fourth order equation (2.1) admits a factorization in the form of (2.3) then the problem of obtaining solutions reduces to solving the second order eqn (2.5). We remark that from $(2.4)_1$ and (2.7) we have that $c(R)$ is given by

$$c(R) = f_0(R)\omega(R)^2. \tag{2.10}$$

In the wider context of self-adjoint differential equations of any order, solutions of the form (2.9) with $c(R)$ given by (2.10) are given in Hill[7] and are obtained by an alternative method to that described here. The reader is referred to this paper for the appropriate generalizations of (2.6) and (2.9) for higher order linear self-adjoint ordinary differential equations.

Finally, in this section we note that we obtain a slightly more general decomposition of fourth order self-adjoint equations from the observation that the fourth order equation

$$D^2[c(R)D^2u] + \gamma D^2u = 0, \tag{2.11}$$

is self-adjoint where γ is any arbitrary constant and D^2 is defined by (2.2). For (2.11) we find that the functions $f_0(R)$, $f_1(R)$ and $f_2(R)$ are given by

$$\begin{aligned} f_0(R) &= a^2c, \\ f_1(R) &= a(ca')' + 2abc + \gamma a, \\ f_2(R) &= b^2c + [a(bc)']' + \gamma b. \end{aligned} \tag{2.12}$$

We note that in this case the general solution (2.9) is not applicable although the problem of solving (2.1) has been reduced to the problem of solving two second order differential equations. In the following sections we give a number of fourth order self-adjoint differential equations which arise in finite elasticity and we show that the decomposition (2.3) gives rise to known solutions of these equations which had been derived previously by a variety of different methods.

3. SIMULTANEOUS INFLATION AND EXTENSION OF A CYLINDRICAL TUBE

In this section we consider the fourth order self-adjoint differential equations which occur for small deformations superimposed upon the simultaneous inflation and extension of a cylindrical tube. If (R, Θ, Z) and (r, θ, z) denote material and spatial cylindrical polar coordinates respectively, then for homogeneous isotropic incompressible hyperelastic materials the deformation (Rivlin[8]) describing the simultaneous inflation and extension of a cylindrical tube is given by

$$r = (\alpha R^2 + \beta)^{1/2}, \quad \theta = \Theta, \quad z = \alpha^{-1}Z, \tag{3.1}$$

where α and β are constants. If for example we consider the stability of a long tube under uniform external pressure then we are led to consider a deformation of the form,

$$\begin{aligned} r &= (\alpha R^2+\beta)^{1/2}+\epsilon u(R)\cos n\Theta, \\ \theta &= \Theta+\epsilon\alpha^{-1/2}v(R)\sin n\Theta, \\ z &= \alpha^{-1}Z, \end{aligned} \tag{3.2}$$

where n is a positive integer, ϵ is some small parameter for which we can neglect terms of order ϵ^2 and higher and $u(R)$ and $v(R)$ are functions of R which depend upon n. The equations for $u(R)$ and $v(R)$ are given in [3] and the reader is referred to this paper for further details. If we use these equations to obtain a single equation for $u(R)$ then we find after a long but straightforward calculation that $u(R)$ satisfies a fourth order self-adjoint differential equation of the form (2.1) where the functions $f_0(R)$, $f_1(R)$ and $f_2(R)$ are given by

$$\begin{aligned} f_0(R) &= (R^2+K)^2\frac{\phi(R)}{R}, \\ f_1(R) &= 3[(R^2+K)\phi'(R)-R\phi(R)]+(n^2-1)\left\{(2R^2+K)(R^2+K)\frac{\phi'(R)}{R^2}-[R^4+(R^2+K)^2]\frac{\phi(R)}{R^3}\right\}, \\ f_2(R) &= (n^2-1)\left\{n^2\frac{\phi(R)}{R}+(R^2+K)^{1/2}\left[\frac{R\phi'(R)}{(R^2+K)^{1/2}}+\frac{(R^2+2K)\phi(R)}{(R^2+K)^{3/2}}\right]'\right\}, \end{aligned} \tag{3.3}$$

where the constant $K=\beta\alpha^{-1}$ and primes denote differentiation with respect to R. The function $\phi(R)$ appearing in (3.3) is the response function of the material evaluated at the initial deformation (3.1) and is defined explicitly in [3]. We remark that the derivation of (3.3) from the equations given in [3] is extremely long and tedious and it would seem reasonable that the self-adjoint form of the differential equation for $u(R)$ could be obtained directly from the underlying variational principle in finite elasticity although the actual procedure for doing this is by no means apparent.

The problem of obtaining solutions for the deformation (3.2) thus reduces to solving (2.4) with $f_0(R)$, $f_1(R)$ and $f_2(R)$ given by (3.3). In general this appears to be difficult although we can deduce the known solutions given in [1] and [3] by this method. For all response functions $\phi(R)$ and $n=1$ we have the following possible solutions of (2.4),

$$a(R)=(R^2+K)^{3/2},\quad b(R)=0,\quad c(R)=\frac{\phi(R)}{R(R^2+K)}, \tag{3.4}$$

and the reader can easily verify that from this factorization and (2.9) we obtain the solutions given in [3]. If for $n\neq 1$ we look for solutions of (2.4) of the form

$$a(R)=(R^2+K)^{3/2},\quad b(R)=-(n^2-1)(R^2+K)^{1/2},\quad c(R)=\frac{\phi(R)}{R(R^2+K)}, \tag{3.5}$$

then we find that this factorization is possible only for the special case of the Varga material so that in this case the response function $\phi(R)$ must take on a particular form. For further details of this material and the resulting solutions the reader is referred to [1]. It may be worthwhile noting that for K zero and $\phi(R)$ equal to a constant (the shear modulus) the standard linear solutions are obtained by either of the factorizations,

$$\begin{aligned} &a(R)=R^3,\quad b(R)=-(n^2-1)R,\quad c(R)=R^{-3}, \\ &a(R)=R^{-1},\quad b(R)=-(n^2-1)R^{-3},\quad c(R)=R^5. \end{aligned} \tag{3.6}$$

If we consider dynamical instability of a long tube under uniform external pressure (see

Wang and Ertepinar[5]) then instead of (3.2) we consider the time dependent deformation

$$\begin{aligned} r &= (\alpha R^2+\beta)^{1/2} + \epsilon\, e^{i\omega t} u(R)\cos n\Theta, \\ \theta &= \Theta + \epsilon\alpha^{-1/2} e^{i\omega t} v(R)\sin n\Theta, \\ z &= \alpha^{-1} Z, \end{aligned} \tag{3.7}$$

where ω, the frequency is a constant depending on n. If in the momentum equations we take into account the inertial terms then we can show that the fourth order equation for $u(R)$ is still self-adjoint where the coefficients in (2.1), $f_0^*(R)$, $f_1^*(R)$ and $f_2^*(R)$ are given by

$$\begin{aligned} f_0^*(R) &= f_0(R), \\ f_1^*(R) &= f_1(R) + \rho\omega^2 \frac{(R^2+K)^2}{R}, \\ f_2^*(R) &= f_2(R) - \rho\omega^2(n^2-1)R, \end{aligned} \tag{3.8}$$

where ρ is the density of the material and the functions $f_0(R)$, $f_1(R)$ and $f_2(R)$ are exactly as given in (3.3). We note that in this case for K zero the fourth order equation for $u(R)$ takes the form (2.11) with the factorization $(3.6)_1$ and with the constant γ is (2.11) equal to $\rho\omega^2/\mu$ where μ the shear modulus is the constant value of the response function $\phi(R)$.

Finally in this section we consider a small axially symmetric deformation superimposed upon (3.1) which varies along the length of the tube. We consider a deformation of the form,

$$\begin{aligned} r &= (\alpha R^2+\beta)^{1/2} + \epsilon u(R)\cos kZ, \\ \theta &= \Theta, \\ z &= \alpha^{-1}Z + \epsilon w(R)\sin kZ, \end{aligned} \tag{3.9}$$

where k is a constant and $u(R)$ and $w(R)$ are functions of R only. Deformations of this type occur in a variety of problems involving the lengthwise buckling of hollow cylinders and for β non-zero they have not been considered previously in the literature. We mention in addition that essentially the same deformation also arises when considering the stability of a hollow cylinder rotating about its axis with uniform angular velocity. In order to illustrate the self-adjoint equations for (3.9) and in order to avoid excessive detail we shall only consider the case of the neo-Hookean material for which the governing equations for $u(R)$ and $w(R)$ can be given quite simply. If we suppose that the usual pressure function $p^*(r, z)$ associated with incompressible materials is given by

$$p^* = \alpha p_0(R) + \epsilon\alpha^{1/2} p(R)\cos kZ, \tag{3.10}$$

where $p_0(R)$ can be obtained from [3] then from the general equations given in Hill[9] we can deduce the following,

$$\begin{aligned} &\frac{(R^2+K)^{1/2}}{R}u' + \frac{u}{(R^2+K)^{1/2}} + \lambda w = 0, \\ p' &= \mu\left\{\frac{R}{(R^2+K)^{1/2}}\left[u'' + \frac{u'}{R} - k^2 u - \frac{u}{R^2}\right] - \frac{K^2 u'}{R^2(R^2+K)^{3/2}}\right\}, \\ -\lambda p &= \mu\left\{w'' + \frac{w'}{R} - k^2 w + \frac{\lambda K^2 u}{R^2(R^2+K)^{3/2}}\right\}, \end{aligned} \tag{3.11}$$

where the constant $\lambda = k\alpha^{3/2}$ and μ is the shear modulus of the neo-Hookean material. The first equation of (3.11) is the condition of incompressibility while the remaining two equations in (3.11) are obtained from the equilibrium equations. If we eliminate $w(R)$ and $p(R)$ from (3.11) then we again obtain a fourth order self-adjoint equation for $u(R)$ where the functions $f_0(R)$,

$f_1(R)$ and $f_2(R)$ are given by

$$f_0(R) = \frac{(R^2+K)}{R},$$
$$f_1(R) = -\lambda^2 R - k^2\frac{(R^2+K)}{R} - \frac{(3R^4-4KR^2-K^2)}{R^3(R^2+K)}, \tag{3.12}$$
$$f_2(R) = \lambda^2 k^2 R + \lambda^2\frac{(R^6+2KR^4+6K^2R^2+2K^3)}{R^3(R^2+K)^2} + k^2\frac{R}{(R^2+K)} - \frac{3R(R^2-4K)}{(R^2+K)^3}.$$

We remark that for K zero the differential equation for $u(R)$ takes the form (2.11) where

$$a(R) = R, \quad b(R) = -(k^2R + R^{-1}), \quad c(R) = R^{-1}, \tag{3.13}$$

while the constant γ in (2.11) is $\lambda^2 - k^2$. This factorization can be shown to give the solutions obtained by Wilkes[10].

Thus for the deformation (3.1) we have shown that certain plane and axially symmetric small superimposed deformations satisfy fourth order self-adjoint differential equations. Moreover these equations can be reduced to solving second order differential equations if appropriate solutions of either (2.4) or (2.12) can be found. These problems are left for the interested reader. In the following section we give the self-adjoint equation which arises for small deformations superimposed upon the symmetrical expansion of a spherical shell.

4. SYMMETRICAL EXPANSION OF A SPHERICAL SHELL

If (R, Θ, Φ) and (r, θ, ϕ) denote material and spatial spherical polar coordinates respectively then for homogeneous isotropic incompressible hyperelastic materials the symmetrical expansion of a spherical shell is described by the deformation (Green and Shield[11])

$$r = (R^3+K)^{1/3}, \quad \theta = \Theta, \quad \phi = \Phi, \tag{4.1}$$

where K is a constant. If we investigate the stability of a spherical shell under uniform external pressure then we consider the axially symmetric deformation

$$r = (R^3+K)^{1/3} + \epsilon u(R)P_n(\cos\Theta),$$
$$\theta = \Theta + \epsilon v(R)\frac{dP_n(\cos\Theta)}{d\Theta}, \tag{4.2}$$
$$\phi = \Phi,$$

where n is a positive integer, P_n is a Legendre function of degree n and $u(R)$ and $v(R)$ are functions of R only which depend on n. From the equations given in either of [2] or [4] we can show that the fourth order equation for $u(R)$ takes the form (2.1) where the functions $f_0(R)$, $f_1(R)$ and $f_2(R)$ are given by

$$f_0(R) = (R^3+K)^2\frac{\phi_1(R)}{R^2},$$
$$f_1(R) = (R^3+K)[4(R^3-K) + n(n+1)(2R^3+K)]\frac{\phi_1'(R)}{2R^3}$$
$$+ [4K(3R^3+K) - n(n+1)(2R^6+2KR^3+K^2)]\frac{\phi_1(R)}{R^4}, \tag{4.3}$$
$$f_2(R) = [n(n+1)-2]\left\{n(n+1)\phi_1(R) + (R^3+K)^{2/3}\left[\frac{R^2\phi_1'(R)}{(R^3+K)^{2/3}} - \frac{2K^2\phi_1(R)}{R^2(R^3+K)^{5/3}}\right]'\right\},$$

ere primes denote differentiation with respect to R and $\phi_1(R)$ is one of the response functions of the material which is evaluated at the initial deformation (4.1) and is defined explicitly in either of [2] or [4]. Again the problem of solving (2.4) with $f_0(R)$, $f_1(R)$ and $f_2(R)$ given by (4.3) is in general difficult. The solutions given in [4] which are applicable for all response functions $\phi_1(R)$ with $n = 1$ are obtained from the factorization,

$$a(R) = \frac{(R^3+K)^{5/3}}{R}, \quad b(R) = 0, \quad c(R) = \frac{\phi_1(R)}{(R^3+K)^{4/3}}. \tag{4.4}$$

We can also show for $n \neq 1$ that the factorization,

$$a(R) = \frac{(R^3+K)^{5/3}}{R}, \quad b(R) = -[n(n+1)-2](R^3+K)^{2/3}, \quad c(R) = \frac{\phi_1(R)}{(R^3+K)^{4/3}}, \tag{4.5}$$

is only meaningful for the particular material considered in [2] and the reader is referred to this paper for further details. We note that for K zero and $\phi_1(R)$ constant the standard linear solutions are obtained from either of the following factorizations,

$$\begin{gathered} a(R) = R^4, \quad b(R) = -[n(n+1)-2]R^2, \quad c(R) = R^{-4}, \\ a(R) = 1, \quad b(R) = -n(n+1)R^{-2}, \quad c(R) = R^4. \end{gathered} \tag{4.6}$$

For both of the deformations (3.2) and (4.2) we have shown that there exist special materials for which the fourth order equations can be readily factorized. It may be of some interest to note that the appropriate factorizations and hence the special materials arise naturally from the following *ad-hoc* considerations. We consider for example the deformation (4.2) for which the pressure function $p^*(r, \theta)$ is given by

$$p^* = p_0(R) + \epsilon p(R) P_n(\cos \Theta), \tag{4.7}$$

where $p_0(R)$ is the pressure function corresponding to (4.1) and is given explicitly in [2]. From [2] we see that the derivative of the function $p(R)$ takes the form,

$$p' = g_0(R)\phi_1 + g_1(R)\phi_1' + g_2(R)\phi_1'', \tag{4.8}$$

where the functions $g_0(R)$, $g_1(R)$ and $g_2(R)$ involve $u(R)$ and its derivatives and can be readily identified from [2]. The condition that the right-hand side of (4.8) be an exact differential expression for $\phi_1(R)$ is

$$g_0(R) = [g_1(R) - g_2'(R)]'. \tag{4.9}$$

If we use the appropriate expressions for the coefficients $g_0(R)$, $g_1(R)$ and $g_2(R)$ then we find that (4.9) becomes

$$\left[\frac{(R^3+K)^{5/3}}{R}u'\right]' - [n(n+1)-2](R^3+K)^{2/3}u = 0, \tag{4.10}$$

and from this equation and $(4.3)_1$ we are led to the factorization (4.5) which as we have already mentioned gives rise to the special material considered in [2]. Evidently this derivation of (4.5) is completely *ad-hoc* although in precisely the same way we can generate the factorization (3.5) for the Varga material for which solutions are given in [1]. In the following section we consider a deformation for which this procedure is not effective. Thus while this approach is not universal, it could perhaps be applicable to other non-homogeneous deformations.

5. PLANE STRAIGHTENING AND STRETCHING OF A SECTOR OF A CIRCULAR-CYLINDRICAL TUBE

If (R, Θ, Z) denote material cylindrical polar coordinates and (x, y, z) denote spatial rectangular cartesian coordinates then for a homogeneous isotropic incompressible hyperelastic material the deformation (Ericksen[12]) describing the plane straightening and stretching of a sector of a circular-cylindrical tube is given by

$$x = AR^2 + B, \quad y = C\Theta + D, \quad z = Z, \tag{5.1}$$

where A, B, C and D are all constants such that $2AC = 1$. If we consider the small superimposed deformation of the form

$$\begin{aligned} x &= AR^2 + B + \epsilon U(R) \cos n\Theta, \\ y &= C\Theta + D + \epsilon V(R) \sin n\Theta, \\ z &= Z, \end{aligned} \tag{5.2}$$

where n is a positive integer and $U(R)$ and $V(R)$ are functions of R only which depend upon n, then we find that the fourth order equation for $U(R)$ becomes self-adjoint if we take R^2 as the independent variable instead of R. We find moreover that it is convenient to define the new independent variable ξ by

$$\xi = nAC^{-1}R^2. \tag{5.3}$$

If we suppose that the pressure function $p^*(x, y)$ is given by

$$p^* = p_0(\xi) + \epsilon p(\xi) \cos n\Theta, \tag{5.4}$$

then from the general equations given in Hill[13] we find that

$$p_0(\xi) = \frac{\xi}{m}\phi(\xi) + \sigma, \tag{5.5}$$

where $m = n/2$, σ is a constant and $\phi(\xi)$ is the response function of the material evaluated at the initial deformation (5.1) and is a function of ξ through the invariant I_0 of (5.1) which can be shown to be given by

$$I_0 = \frac{\xi}{m} + \frac{m}{\xi}. \tag{5.6}$$

The reader is referred to [13] for the precise definitions of the response function $\phi(\xi)$ and the invariant I_0. With the notation

$$u(\xi) = U(R), \quad v(\xi) = V(R), \tag{5.7}$$

we find from the general equations given in [13] that the incompressibility condition and equations of equilibrium become respectively,

$$\begin{aligned} &u' + v = 0, \\ &\frac{C}{2}p' = \phi\left(\xi\ddot{u} + 2\dot{u} - m^2\frac{u}{\xi}\right) + 2\phi'(\xi^2\ddot{u} + 3\xi\dot{u}) + 2\phi''\xi^2\dot{u}, \\ &-\frac{C}{2}p = \phi\left(\xi\ddot{v} + \dot{v} - m^2\frac{v}{\xi} - u\right) + \phi'(\xi\dot{v} - 2m^2\dot{u} - \xi u), \end{aligned} \tag{5.8}$$

where primes denote differentiation with respect to ξ. If we eliminate $v(\xi)$ and $p(\xi)$ from (5.8) then we obtain a fourth order self-adjoint equation for $u(\xi)$ where the functions $f_0(\xi)$, $f_1(\xi)$ and

$f_2(\xi)$ are given by

$$f_0(\xi) = \xi\phi(\xi),$$
$$f_1(\xi) = 2(m^2 - \xi^2)\phi'(\xi) - \left(\frac{m^2}{\xi} + \xi\right)\phi(\xi), \tag{5.9}$$
$$f_2(\xi) = [\xi\phi(\xi)]'' + \frac{m^2}{\xi}\phi(\xi).$$

We note that for $m = 1/2$ ($n = 1$) $u(\xi) = \xi^{1/2}$ is a solution of the self-adjoint equation and that this solution arises from the invariance of the governing equations under the translation

$$(X, Y, Z) \rightarrow (X + \epsilon, Y, Z), \tag{5.10}$$

where (X, Y, Z) denote material rectangular cartesian coordinates.

Finally in this section we remark that there appears to be no special material for (5.2) for which the fourth order equation is readily factorized. From the condition (4.9) and $(5.8)_2$ we obtain

$$u'' + \frac{m^2}{\xi^2}u = 0, \tag{5.11}$$

so that we are led to consider the factorization

$$a(\xi) = 1, \quad b(\xi) = \frac{m^2}{\xi^2}, \quad c(\xi) = \xi\phi(\xi). \tag{5.12}$$

From (2.4) and (5.9) we find that this factorization is only meaningful for $m = 1/2$ and hence in this case, the procedure noted in the previous section is not effective.

6. CONCLUSION

We have shown that particular fourth order ordinary differential equations arising for small deformations superimposed upon large non-homogeneous deformations of homogeneous isotropic incompressible hyperelastic materials are self-adjoint. This property has not been noted previously and while it does not lead immediately to solutions, it does mean that many standard results in analysis are applicable to these equations. Moreover the derivation of these equations is extremely long and tedious and the property of self-adjointness does provide some symmetry to what are otherwise complicated problems. We have also shown that in principle every fourth order self-adjoint differential equation can be factored in terms of a single second order self-adjoint operator. In practice however the determination of this factorization is in general difficult although this observation and that of self-adjointness does permit a unified derivation of existing solutions of these equations. The outstanding problems of deriving these self-adjoint equations directly from the variational principle in finite elasticity and of obtaining the appropriate factorizations for the equations given here are left for the interested reader.

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